लक्ष्यं विश्वमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2023

MTMACOR09T-MATHEMATICS (CC9)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: $2\times 5=10$

(a) If S be the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying the inequality $x+y+z<1$, determine whether or not S is open. 2

(b) Is the set $\mathbb{R}^n$ open? Justify your answer. 2

(c) Find the closure of $\{(x, y): 1<x^2+y^2<2\}$. 2

(d) When a rational function $f(x)=\dfrac{P(x)}{Q(x)}$ (where P, Q are polynomials in the components of x) is continuous at each point x? 2

(e) Show that the function $f(x,y)=|x|+|y|$, $(x, y)\in\mathbb{R}^2$ possesses an extreme value at (0, 0) although $f_x(0,0)$, $f_y(0,0)$ do not exist. 2

(f) Find the gradient vector at each point at which it exists for the scalar field defined by $f(x, y)=x^2+y^2\sin(xy)$. 2

(g) Find $\iint\limits_R x^2dxdy$ where R is the region bounded by $x=0$, $y=0$ and $y=\cos x$. 2

(h) Use Green's theorem to compute the work done by the force field $f(x, y)=(y+3x)i+(2y-x)j$ in moving a particle once around the ellipse $4x^2+y^2=4$ in the counterclockwise. 2

2. (a) Show that the function is discontinuous at (0, 0) 4

$$f(x, y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & \text{when } x\neq y\\ 0\ , & x=y\end{cases}$$

(b) If $f(x, y)$ is continuous at (a, b) and $f(a, b)\neq 0$ then prove that there exists a neighbourhood of (a, b) where $f(x, y)$ and $f(a, b)$ maintain the same sign. 4

3. (a) The scalar field is defined by 1+1+1+1

$$f(x, y)=\begin{cases}3y, & \text{when } x=y\\ 0\ , & \text{otherwise}\end{cases}$$

Do the partial derivatives $D_1f(0,0)$ and $D_2f(0,0)$ exist? If exist find their values. Find the directional derivative at the origin in the direction of the vector $i+j$.

(b) Evaluate $\iint\limits_R (x+2y)\,dxdy$, over the rectangle $R=[1, 2, 3, 5]$. 4

4. (a) Show that if $xyz = a^2(x+y+z)$, then the minimum value of $xy+zx+zy$ is $9a^2$. 4

(b) A function f is defined on the rectangle $R=[0,1;0,1]$ as follows: 4

$$f(x,y)=\begin{cases} \frac{1}{2}, & \text{when } y \text{ is rational} \\ x, & \text{when } y \text{ is irrational} \end{cases}$$

Show that the double integral $\iint_R f(x,y)\,dxdy$, does not exist.

5. (a) If $lx+my+nz=1$, l, m, n are positive constants, show that the stationary value of $xy+yz+zx$ is $(2lm+2mn+2nl-l^2-m^2-n^2)^{-1}$. 4

(b) For the vector field $F(x,y,z)=(x^2+yz)i+(y^2+xz)j+(z^2+xy)k$ compute the curl and divergence. 2+2

6. (a) Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$ (using Lagrange's method of multiplier). 4

(b) Let $y=F(x,t)$, where F is a differentiable function of two independent variables x and t which are related to two variables u and v by the relations $u=x+ct$, $v=x-ct$ ($c=\text{constant}\neq 0$). Prove that the partial differential equation $\frac{\partial^2 y}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}=0$ can be transformed into $\frac{\partial^2 y}{\partial u \partial v}=0$. 4

7. (a) Evaluate $\iint_E (x^2+y^2)\,dxdy$ over the region E bounded by $xy=1$, $y=0$, $y=x$, $x=2$. 4

(b) Show that $\iiint_E z^2\,dxdydz$, where E is the region of the hemisphere $z\geq 0$, $x^2+y^2+z^2\leq a^2$, is $\frac{2}{15}\pi a^5$. 4

8. (a) Show that the entire volume bounded by the positive side of the three co-ordinate planes and the surface $\left(\frac{x}{a}\right)^{1/2}+\left(\frac{y}{b}\right)^{1/2}+\left(\frac{z}{c}\right)^{1/2}=1$ is $\frac{abc}{90}$. 4

(b) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ around the triangle OPQ whose vertices are $O(0,0,0)$, $P(2,0,0)$ and $Q(2,1,1)$, where $\vec{F}=(2x^2+y^2)\hat{i}+(3y-4z)\hat{j}+(x-y+z)\hat{k}$. 4

9. (a) Using Stokes' theorem, evaluate $\oint_C (xydx+xy^2dy)$, where C is the square in the xy-plane with vertices $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$. 4

(b) Using Green's theorem, evaluate $\int_\Gamma \{(y-\sin x)\,dx+\cos x\,dy\}$ where Γ is the triangle enclosed by the lines $y=0$, $x=\pi$ and $y=\frac{2x}{\pi}$. 4

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